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Piecewise quartic polynomial curves with a local shape parameter☆

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Abstract

Piecewise quartic polynomial curves with a local shape parameter are presented in this paper. The given blending function is an extension of the cubic uniform B-splines. The changes of a local shape parameter will only change two curve segments. With the increase of the value of a shape parameter, the curves approach a corresponding control point. The given curves possess satisfying shape-preserving properties. The given curve can also be used to interpolate locally the control points with GC^2 continuity. Thus, the given curves unify the representation of the curves for interpolating and approximating the control polygon. As an application, the piecewise polynomial curves can intersect an ellipse at different knot values by choosing the value of the shape parameter. The given curve can approximate an ellipse from the both sides and can then yield a tight envelope for an ellipse. Some computing examples for curve design are given.

Keywords: Spline curve; Polynomial curve; Local interpolation; Shape parameter

1. Introduction

Piecewise cubic B-spline curves with four consecutive control points for each curve segment are simple, flexible and can be used conveniently; see [10]. However, the position of piecewise cubic B-spline curves is fixed relative to their control polygon. The control polygon must be changed if the shapes of the curves need to be adjusted. Although the weight numbers in the cubic rational B-spline curves, see [5,13], have an influence on adjustment of the shape of the curves, the changes of a weight number will

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change four curve segments. For piecewise rational Bézier curves, we also need continuity conditions on the control points. If knots are considered with multiplicity, for example, if quartic B-splines with double knots are considered, then the B-splines have C^2 continuity at knots and have twice as many control points as splines with single knots. Thus, although a weight number or a control point of the quartic non-uniform B-spline curves only affects two or three segments (not five segments), a curve segment has five control points and five weight numbers. For given control points, changing the weight numbers to adjust the shape of a curve is quite opaque to the user.

In geometric modelling and computer graphics, in order to improve the shape of a curve and adjust the extent to which a curve approaches its control polygon, some methods of generating curves are presented by using tension parameters; see [1,2,8,9,11,12,14,15]. In [2], the cubic β-curve is one of the important piecewise polynomial curves. However, the shape parameters of the β-curves are global parameters. If and only if the shape parameters $\beta_1 = 1$ and $\beta_2 = 0$, are the β-curves C^2 continuous for a uniform knot sequence. In other words, the shape of the β-curves cannot be changed in this case. We also need a proper choice of the shape parameters for a desired terminal property of the β-curves. In order to introduce local shape parameters into β-curves, a restricted form of a quintic Hermite interpolation is used for β parameters in [1]. Thus, the degree of the β-curves itself is raised. In [12], a complicated explicit quartic β-spline with shape parameters is presented, and five control points are used for each curve segment. The C-B spline curve in [14,15] is another kind of curve with a shape parameter. The C-B spline curves are constructed with trigonometric functions and they can only be situated on one side of the cubic B-spline curves. The cubic B-spline curves are closer to their control polygon than the C-B spline curves.

The purpose of this paper is to present methods of generating curves with four control points for each curve segment and with local shape parameters. We make efforts to set shape parameters so that a shape parameter affects curve segments as little as possible and has a predictable adjusting role on the curves.

Polynomial approximation methods for conic sections have been discussed recently in some researches; see [3,4,6,7]. The method in [7] shows that a considerably better approximation from one side of the conic sections can be achieved. The method uses three control points for each curve segment and induces the polynomial curve to intersect a conic section at the ends and midpoint of a parameter interval. The degree of the given polynomial curves in [7] is an odd number. As an application, we will show that the quartic polynomial curves in this paper approximate an ellipse well. By using the shape parameter, our method will induce the quartic polynomial curve and an ellipse to intersect at different knot values.

2. Piecewise quartic polynomial curves

2.1. The construction of the curves

Definition 1. Let $\lambda_i \in [-8, 1]$, $i = 1, 2, \ldots, n-1$. For $t \in [0, 1]$, $i = 1, 2, \ldots, n-2$, the functions

$$b_{i0}(t) = \frac{1}{24}[(4-\lambda_i)(1-t)^4 + 4(1-\lambda_i)(1-t)^3 t], \tag{1}$$

$$\begin{aligned} b_{i1}(t) = \frac{1}{24}[&2(8+\lambda_i)(1-t)^4 + 8(8+\lambda_i)(1-t)^3 t + 72(1-t)^2 t^2 \\ &+ 4(7-\lambda_{i+1})(1-t)t^3 + (4-\lambda_{i+1})t^4], \end{aligned} \tag{2}$$

$$b_{i2}(t) = \tfrac{1}{24}[(4-\lambda_i)(1-t)^4 + 4(7-\lambda_i)(1-t)^3 t + 72(1-t)^2 t^2 + 8(8+\lambda_{i+1})(1-t)t^3 + 2(8+\lambda_{i+1})t^4], \tag{3}$$

$$b_{i3}(t) = \tfrac{1}{24}[4(1-\lambda_{i+1})(1-t)t^3 + (4-\lambda_{i+1})t^4] \tag{4}$$

are called blending functions with parameter sequence $\{\lambda_k\}$.

It can be verified that the functions $b_{ij}(t)$ $(j=0, 1, 2, 3)$ are cubic uniform B-splines when $\lambda_i = \lambda_{i+1} = 0$; see [10].

Definition 2. Given control points $P_i \in \mathbb{R}^d$ $(d = 2, 3; i = 0, 1, \ldots, n)$ and knots $u_1 < u_2 < \cdots < u_{n-1}$, for $u \in [u_i, u_{i+1}]$, $i = 1, 2, \ldots, n-2$, the curves

$$C(u) = \sum_{j=0}^{3} b_{ij}(t) P_{i+j-1} \tag{5}$$

are defined to be piecewise polynomial curves, where $t = (u - u_i)/h_i$, $h_i = u_{i+1} - u_i$.

Obviously, curves $C(u)(\lambda_i \neq 0)$ are piecewise quartic polynomial curves defined on $[u_1, u_{n-1}]$. The parameter λ_i is a local parameter. Curves $C(u)$ are cubic uniform B-spline curves when all $\lambda_i = 0$. Therefore, (5) is an extension of the cubic uniform B-spline curves. When all h_i are the same, the $C(u)$ are uniform polynomial curves.

2.2. Parametric continuity

Curves (5) are piecewise polynomial curves. We need to show the continuity of the curves.

Theorem 1. *For $u \in [u_1, u_{n-1}]$, curves (5) are GC^2 continuous. The uniform curves (5) are C^2 continuous.*

Proof. For $i = 1, 2, \ldots, n-2$, we have

$$C(u_i^+) = \tfrac{1}{24}[(4-\lambda_i)P_{i-1} + 2(8+\lambda_i)P_i + (4-\lambda_i)P_{i+1}], \tag{6}$$

$$C(u_{i+1}^-) = \tfrac{1}{24}[(4-\lambda_{i+1})P_i + 2(8+\lambda_{i+1})P_{i+1} + (4-\lambda_{i+1})P_{i+2}], \tag{7}$$

$$C'(u_i^+) = \frac{1}{2h_i}(P_{i+1} - P_{i-1}), \tag{8}$$

$$C'(u_{i+1}^-) = \frac{1}{2h_i}(P_{i+2} - P_i), \tag{9}$$

$$C''(u_i^+) = \frac{2+\lambda_i}{2h_i^2}(P_{i-1} - 2P_i + P_{i+1}), \tag{10}$$

$$C''(u_{i+1}^-) = \frac{2+\lambda_{i+1}}{2h_i^2}(P_i - 2P_{i+1} + P_{i+2}). \tag{11}$$

Thus, we obtain

$$C^{(k)}(u_i^-) = \left(\frac{h_i}{h_{i-1}}\right)^k C^{(k)}(u_i^+), \quad k=0,1,2; \quad i=2,3,\ldots,n-2. \tag{12}$$

This implies the theorem. □

From (8) and (9), we know that the tangent line of curves $C(u)$ at the point $C(u_i)$ is parallel to the line segment $P_{i-1}P_{i+1}$ (for any λ_i). This property corresponds to the property of the cubic B-spline curves.

Theorem 2. *The curvature of the curves $C(u)$ at $u = u_i$ is*

$$K(u_i) = 4|\lambda_i + 2| \frac{\|(P_i - P_{i-1}) \times (P_{i+1} - P_i)\|}{\|P_{i+1} - P_{i-1}\|^3}. \tag{13}$$

Proof. According to (8) and (10), the curvature of the curves $C(u)$ at $u = u_i$ is

$$\begin{aligned} K(u_i) &= \frac{\|C'(u_i) \times C''(u_i)\|}{\|C'(u_i)\|^3} \\ &= 2|\lambda_i + 2| \frac{\|(P_{i+1} - P_{i-1}) \times (P_{i-1} - 2P_i + P_{i+1})\|}{\|P_{i+1} - P_{i-1}\|^3} \\ &= 4|\lambda_i + 2| \frac{\|(P_i - P_{i-1}) \times (P_{i+1} - P_i)\|}{\|P_{i+1} - P_{i-1}\|^3}. \qquad \square \end{aligned}$$

According to (13), the local shape parameter λ_i controls the curvature of the curves $C(u)$ at the end of the curve segments. When $\lambda_i > -2$, the curvature of the curves at $u = u_i$ increases with the increase of λ_i. When $\lambda_i < -2$, the curvature of the curves at $u = u_i$ increases with the decrease of λ_i.

In order to construct closed curves, we can periodically assign control points by setting $P_{n+1} = P_0$, $P_{n+2} = P_1$, $P_{n+3} = P_2$, and expand the knots by setting $u_{n+2} > u_{n+1} > u_n > u_{n-1}$, and let $\lambda_i \in [-8, 1]$, $i = n, n+1, n+2$, $\lambda_{n+2} = \lambda_1$ (such that $C(u_1) = C(u_{n+2})$, $h_1 C'(u_1^+) = h_{n+1} C'(u_{n+2}^-)$, $h_1^2 C''(u_1^+) = h_{n+1}^2 C''(u_{n+2}^-)$). Thus, the parametric formulae for closed curves are defined on the interval $[u_1, u_{n+2}]$.

For open curves, we can also expand the curve segment by setting $\lambda_0, \lambda_n \in [-8, 1]$, $u_0 < u_1$, $u_n > u_{n-1}$, $P_{-1} = 2P_0 - P_1$, $P_{n+1} = 2P_n - P_{n-1}$. This assures that original points P_0 and P_n are the points on the curves, i.e., $C(u_0) = P_0$, $C(u_n) = P_n$.

Fig. 1 shows open curves and closed curves with all $\lambda_i = 0.5$ (solid line) and on setting a $\lambda_i = 0$ (dashed lines).

2.3. Convex hull properties

Theorem 3. *The blending functions* (1)–(4) *have the following properties*:

(a) *For* $u \in [u_i, u_{i+1}]$, $\sum_{j=0}^{3} b_{ij}(t) = 1$; *and*
(b) *For* $u \in [u_i, u_{i+1}]$, $b_{ij}(t) \geqslant 0$ $(j = 0, 1, 2, 3)$ *if and only if* $\lambda_i, \lambda_{i+1} \in [-8, 1]$.

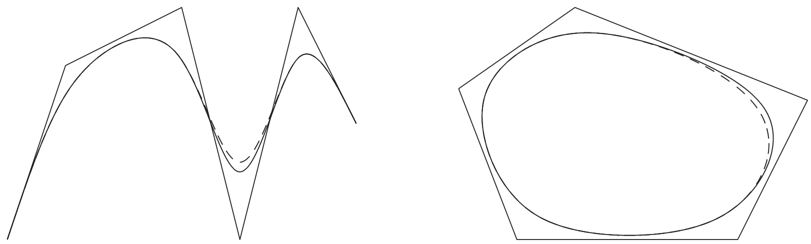

Fig. 1. Open and closed polynomial curves.

Proof. A routine computation yields that $\sum_{j=0}^{3} b_{ij}(t) = 1$.

For $u \in [u_i, u_{i+1}]$, we have $t \in [0, 1]$. Obviously, $b_{i0}(t) \geqslant 0$ if and only if $\lambda_i \leqslant 1$. $b_{i1}(t) \geqslant 0$, $b_{i2}(t) \geqslant 0$ if and only if $\lambda_i \geqslant -8$, $\lambda_{i+1} \geqslant -8$. $b_{i3}(t) \geqslant 0$ if and only if $\lambda_{i+1} \leqslant 1$. $\square$

According to Theorem 3, we know that the curve $C(u)$ is situated in the convex hull of the points $P_{i-1}, P_i, P_{i+1}, P_{i+2}$ for $u \in [u_i, u_{i+1}]$. Moreover, for $u \in [u_i, u_{i+1}]$, we have

$$\begin{aligned} C(u) = {} & [(1-t)^4 + \frac{4-4\lambda_i}{4-\lambda_i}(1-t)^3 t] C(u_i) \\ & + \left[\frac{8+\lambda_i}{4-\lambda_i}(1-t)^3 t + 3(1-t)^2 t^2 + (1-t)t^3\right] P_i \\ & + \left[(1-t)^3 t + 3(1-t)^2 t^2 + \frac{8+\lambda_{i+1}}{4-\lambda_{i+1}}(1-t)t^3\right] P_{i+1} \\ & + \left[\frac{4-4\lambda_{i+1}}{4-\lambda_{i+1}}(1-t)t^3 + t^4\right] C(u_{i+1}). \end{aligned} \tag{14}$$

We also have

$$\begin{aligned} C(u) = {} & (1-t)^4 C(u_i) + 2(1-t)^3 t A_i + 2[(1-t)^3 t + 3(1-t)^2 t^2 + (1-t)t^3] P_{i+\frac{1}{2}} \\ & + 2(1-t)t^3 A_{i+1} + t^4 C(u_{i+1}), \end{aligned} \tag{15}$$

where $A_i = [(1-\lambda_i)P_{i-1} + 2(5+\lambda_i)P_i + (1-\lambda_i)P_{i+1}]/12$, $P_{i+1/2} = (P_i + P_{i+1})/2$. From (14), we know that the curve $C(u)$ lies in the convex hull of the points $C(u_i)$, P_i, P_{i+1}, $C(u_{i+1})$ for $u \in [u_i, u_{i+1}]$. From (15), we know that the curve $C(u)$ lies in the convex hull of the points $C(u_i)$, A_i, $P_{i+1/2}$, A_{i+1}, $C(u_{i+1})$ for $u \in [u_i, u_{i+1}]$. Since λ_i affects $C(u_i)$ and A_i, λ_{i+1} affects $C(u_{i+1})$ and A_{i+1}, from (14) and (15), we clearly know the convex hull in which the curve lies for different λ_i and λ_{i+1}.

2.4. Shape-preserving properties

Shape-preserving properties are important properties in geometric design and computing. If the tangent vector $C'(u)$ is a nonnegative linear combination of the set $\{P_{i+1} - P_i\}$ and the vector $C''(u)$ is a nonnegative linear combination of the set $\{P_{i-1} - 2P_i + P_{i+1}\}$, we regard curve $C(u)$ as shape preserving.

For $u \in [u_i, u_{i+1}]$, $t = (u - u_i)/h_i$, $i = 1, 2, \ldots, n-1$, from (5), we have

$$\begin{aligned}2h_i C'(u) = {} & [(1-t)^3 + (1-\lambda_i)(1-t)^2 t](P_i - P_{i-1}) \\ & + [(1-t)^3 + (5+\lambda_i)(1-t)^2 t + (5+\lambda_{i+1})(1-t)t^2 + t^3](P_{i+1} - P_i) \\ & + [(1-\lambda_{i+1})(1-t)t^2 + t^3](P_{i+2} - P_{i+1}),\end{aligned}$$

$$\begin{aligned}2h_i^2 C''(u) = {} & [(2+\lambda_i)(1-t)^2 + 2(1-\lambda_i)(1-t)t](P_{i-1} - 2P_i + P_{i+1}) \\ & + [2(1-\lambda_{i+1})(1-t)t + (2+\lambda_{i+1})t^2](P_i - 2P_{i+1} + P_{i+2}).\end{aligned}$$

Therefore, for all $\lambda_i \in [-2, 1]$, curve $C(u)$ satisfies shape-preserving properties.

2.5. Local adjustable properties

In this section, we discuss the change of the curve with the change of the parameter λ_i. The following theorem gives the error between the given polynomial curve $C(u)$ and its control polygon.

Theorem 4. *Let $\Delta^2 P_i = P_i - 2P_{i+1} + P_{i+2}$, $P_{i,t} = (1-t)P_i + tP_{i+1}$. For $u \in [u_i, u_{i+1}]$, we have*

$$\|C(u) - P_{i,t}\| \leqslant f(t) \max\{\|C(u_k) - P_k\| : k = i, i+1\}, \tag{16}$$

where $t = (u - u_i)/h_i$,

$$f(t) = 1 - \frac{12}{4-\lambda_i}(1-t)^3 t - 6(1-t)^2 t^2 - \frac{12}{4-\lambda_{i+1}}(1-t)t^3 \leqslant 1.$$

Proof. According to (5), we have

$$C(u) = b_{i0}(t)\Delta^2 P_{i-1} + P_{i,t} + b_{i3}(t)\Delta^2 P_i. \tag{17}$$

From this, we obtain

$$C(u) - P_{i,t} = \frac{24}{4-\lambda_i} b_{i0}(t)(C(u_i) - P_i) + \frac{24}{4-\lambda_{i+1}} b_{i3}(t)(C(u_{i+1}) - P_{i+1}).$$

Let $f(t) = 24b_{i0}(t)/(4-\lambda_i) + 24b_{i3}(t)/(4-\lambda_{i+1})$; then

$$\begin{aligned}0 < f(t) &= (1-t)^4 + \frac{4(1-\lambda_i)}{4-\lambda_i}(1-t)^3 t + \frac{4(1-\lambda_{i+1})}{4-\lambda_{i+1}}(1-t)t^3 + t^4 \\ &= 1 - \frac{12}{4-\lambda_i}(1-t)^3 t - 6(1-t)^2 t^2 - \frac{12}{4-\lambda_{i+1}}(1-t)t^3 \leqslant 1.\end{aligned}$$

Thus, we obtain (16) immediately. □

Obviously, $f(t)$ is decreasing on λ_i or λ_{i+1}. Inequality (16) shows that the upper bounds of $\|C(u) - P_{i,t}\|$ decrease as λ_i and λ_{i+1} increase. Further, increasing λ_i decreases $b_{i0}(t)$, $b_{i2}(t)$ and increases $b_{i1}(t)$ for all $t \in (0, 1)$. Since $\sum_{j=0}^{3} b_{ij}(t) = 1$ and $b_{i3}(t)$ is independent of λ_i, we know that the curve $C(u)$ approaches the point P_i with the increase of λ_i.

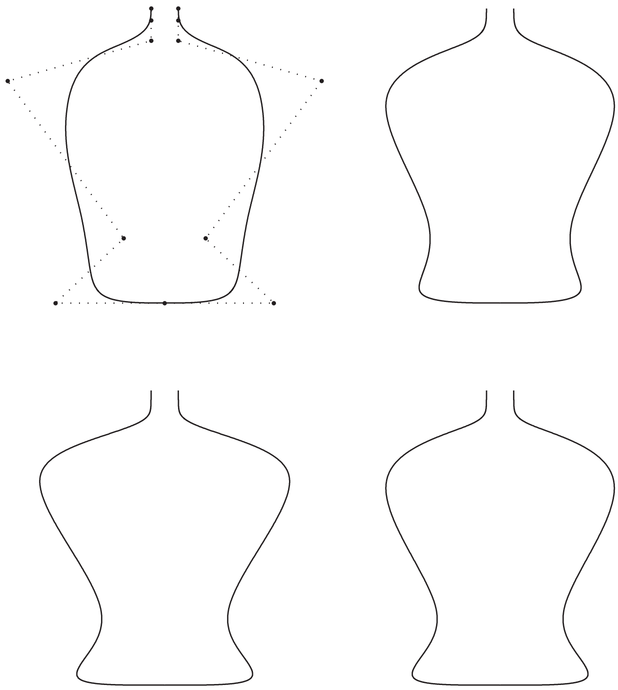

Fig. 2. Examples of curve design with different values of λ_i.

Therefore, the λ_i parameter serves to locally control the tension in the curves: parameter λ_i only affects two curve segments $C(u)(u \in (u_{i-1}, u_{i+1}))$ and controls the position of the curves $C(u)$ in a corner of the control polygon. Increasing λ_i induces the curves $C(u)(u \in (u_{i-1}, u_{i+1}))$ to move locally toward the control vertex P_i. As λ_i and λ_{i+1} increase, the curve segment $C(u)(u \in [u_i, u_{i+1}])$ moves toward the edge $P_i P_{i+1}$ of the control polygon.

Fig. 2 shows an example of curve design by setting all $\lambda_i = -1.5, 0, 1$, respectively, and all $\lambda_i = 1$ except two $\lambda_i = 0$ in the last figure.

Now we discuss the error between the given polynomial curve and the corresponding uniform B-spline curve.

Theorem 5. *Let $B(u) = C(u)$ when all $\lambda_i = 0$. Then, for $u \in [u_i, u_{i+1}]$, we have*

$$\|C(u) - B(u)\| \leqslant \tfrac{1}{24} \max\{\|\lambda_k \Delta^2 P_{k-1}\| : k = i, i+1\}. \tag{18}$$

Proof. From (17), we have

$$B(u) - C(u) = \frac{\lambda_i}{24}(1-t)^3(1+3t)\Delta^2 P_{i-1} + \frac{\lambda_{i+1}}{24} t^3(4-3t)\Delta^2 P_i.$$

Since $(1-t)^3(1+3t) + t^3(4-3t) = 1 - 6(1-t)^2 t^2 \leqslant 1$, we obtain (18). □

Inequality (18) shows that the upper bounds of $\|C(u) - B(u)\|(u \in [u_i, u_{i+1}])$ decrease as λ_i and λ_{i+1} tend to zero.

2.6. Local interpolation

Curve (5) can also be used for local interpolation. Let $\lambda_i = 4$; from (6) and (7), we have $C(u_i) = P_i (1 \leqslant i \leqslant n-1)$. This means that curve $C(u)$ interpolates point P_i at $u=u_i$ locally. Thus, we provide a GC^2 continuous local interpolation method without solving a linear system or any additional control points. The given piecewise quartic polynomial curves unify the representation of the curves for interpolating and approximating the control polygon.

Now we will show that the local interpolation curve preserves the shape of the control polygon well.

Consider the curve segment $C(u)$ $(u \in [u_i, u_{i+1}])$. Let $\lambda_i = \lambda_{i+1} = 4$; we have $C(u_i) = P_i$, $C(u_{i+1}) = P_{i+1}$, and

$$h_i C'(u) = \sum_{j=0}^{2} a_{ij}(t)(P_{i+j} - P_{i+j-1}),$$

where $t = (u - u_i)/h_i \in [0, 1]$,

$$\begin{aligned} a_{i0}(t) &= \tfrac{1}{2}(1-t)^2(1-4t), \\ a_{i1}(t) &= \tfrac{1}{2}[1 + 6(1-t)t], \\ a_{i2}(t) &= \tfrac{1}{2}(4t-3)t^2. \end{aligned}$$

Obviously, we have

$$C'(u_i) = \frac{1}{2h_i}(P_{i+1} - P_{i-1}), \quad C'(u_{i+1}) = \frac{1}{2h_i}(P_{i+2} - P_i).$$

Since $a_{i1}(t)$ increases as t approaches $1/2$, and

$$\sum_{j=0}^{2} a_{ij}(t) = 1,$$
$$a_{i1}(t) - (a_{i0}(t) + a_{i2}(t)) = 6(1-t) \quad t \geqslant 0,$$

we can deduce that $C'(u)$ approaches $(P_{i+1} - P_i)/h_i$ as u approaches $(u_{i+1} + u_i)/2$.

When $\lambda_i = \lambda_{i+1} = 4$, we also have

$$h_i^2 C''(u) = 3(1-2t)[(1-t)(P_{i-1} - 2P_i + P_{i+1}) - t(P_i - 2P_{i+1} + P_{i+2})].$$

Therefore, we have

$$C''(u_i) = \frac{3}{h_i^2}(P_{i-1} - 2P_i + P_{i+1}), \quad C''(u_{i+1}) = \frac{3}{h_i^2}(P_i - 2P_{i+1} + P_{i+2}).$$

Fig. 3 shows a local planar interpolation curve with $\lambda = (0, 4, 0, 0, 4, 4, 4, 4, 4, 0, 0)$ on the left and a space interpolation curve with $\lambda = (0, 4, 1, 1, 4, 0, 1, 4, 0)$ on the right.

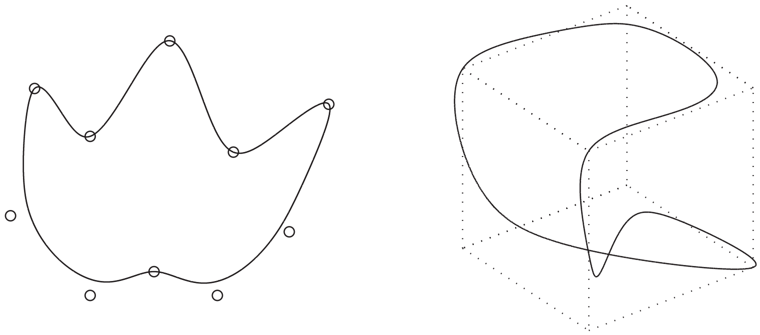

Fig. 3. The local planar and space interpolation curves.

3. The properties of approximation ellipses

It is well known that polynomial curves cannot express ellipses. The polynomial representation of a curve is important. In this section, as an application, we show that curves (5) can approximate ellipses well by choosing the value of the shape parameter.

For brevity, we consider the ellipse

$$\frac{x^2}{\alpha^2}+\frac{y^2}{\beta^2}=1, \quad \alpha, \beta>0. \tag{19}$$

Corresponding to this, we write the approximation curves as

$$C(u)=\sum_{j=0}^{3} b_{ij}(t)P_{i+j-1}, \quad u\in[u_i,u_{i+1}], \ \ i=1,2,3,4 \tag{20}$$

where $t=(u-u_i)/h_i$, h_i and $\lambda_i=\lambda$ are constants. We need to determine the control points P_j $(j=0,1,\ldots,6)$ and the shape parameter λ.

Let $Q_0=(-a,-b)$, $Q_1=(a,-b)$, $Q_2=(a,b)$, $Q_3=(-a,b)$,

$$P_j=Q_{j-4[\frac{j}{4}]}, \quad j=0,1,\ldots,6.$$

For $i\in\{1,2,3,4\}$, let $(x,y)=C(u)$. A routine computation gives that

$$\begin{aligned}72\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}\right)&=(8+\lambda)^2+2(1-t)^2t^2[9\lambda^2(1-t)^2t^2\\&\quad-4(\lambda^2+13\lambda+4)(1-t)t-6(\lambda^2+10\lambda+4)].\end{aligned} \tag{21}$$

Theorem 6. *For $i\in\{1,2,3,4\}$, $s_0\in[0,1/4]$, let $(x,y)=C(u)$,*

$$\lambda=\frac{2}{9s_0^2-4s_0-6}\left(13s_0+15-3\sqrt{4s_0^3+23s_0^2+38s_0+21}\right). \tag{22}$$

We have $\sqrt{21}-5\leqslant\lambda<0$ *and*

$$\frac{72}{(8+\lambda)^2}\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}\right)=1+f(s)(1-t)^2t^2\left(t-\frac{1-\sqrt{1-4s_0}}{2}\right)\left(t-\frac{1+\sqrt{1-4s_0}}{2}\right), \tag{23}$$

where $s=(1-t)t$,

$$f(s)=\frac{2}{(8+\lambda)^2}[4(\lambda^2+13\lambda+4)-9\lambda^2(s+s_0)]. \tag{24}$$

Proof. For $s_0\in[0,1/4]$, it is easy to show that $\sqrt{21}-5\leqslant\lambda<0$. Since

$$9\lambda^2s_0^2-4(\lambda^2+13\lambda+4)s_0-6(\lambda^2+10\lambda+4)=0,$$

according to (21), we have

$$\begin{aligned}\frac{72}{(8+\lambda)^2}\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}\right)-1&=\frac{2s^2}{(8+\lambda)^2}[9\lambda^2s^2-4(\lambda^2+13\lambda+4)s-6(\lambda^2+10\lambda+4)]\\&=\frac{2s^2}{(8+\lambda)^2}[4(\lambda^2+13\lambda+4)-9\lambda^2(s+s_0)](s_0-s).\end{aligned}$$

This implies (23). □

According to Theorem 6, if we let

$$a=\frac{6\sqrt{2}}{8+\lambda}\alpha,\quad b=\frac{6\sqrt{2}}{8+\lambda}\beta,$$

then ellipse (19) and each curve segment of (20) intersect at $t=0,1,(1\pm\sqrt{1-4s_0})/2$. Obviously, $(1-\sqrt{1-4s_0})/2\in[0,1/2]$, $(1+\sqrt{1-4s_0})/2\in[1/2,1]$. Particularly, we have the following:

(a) When $s_0=0$, we have $\lambda=\sqrt{21}-5\approx-0.4174$,

$$\begin{aligned}\frac{x^2}{\alpha^2}+\frac{y^2}{\beta^2}-1=&-f(s)(1-t)^3t^3,\\&-0.1879<\frac{3(23\lambda+6)}{(8+\lambda)^2}\leqslant f(s)\leqslant\frac{24\lambda}{(8+\lambda)^2}<-0.1742.\end{aligned}$$

(b) When $s_0=1/4$, we have $\lambda=\frac{8(48\sqrt{2}-73)}{103}\approx-0.3975$,

$$\begin{aligned}\frac{x^2}{\alpha^2}+\frac{y^2}{\beta^2}-1=&f(s)(1-t)^2t^2\left(t-\frac{1}{2}\right)^2,\\&-0.1644<f(s)<-0.1520.\end{aligned}$$

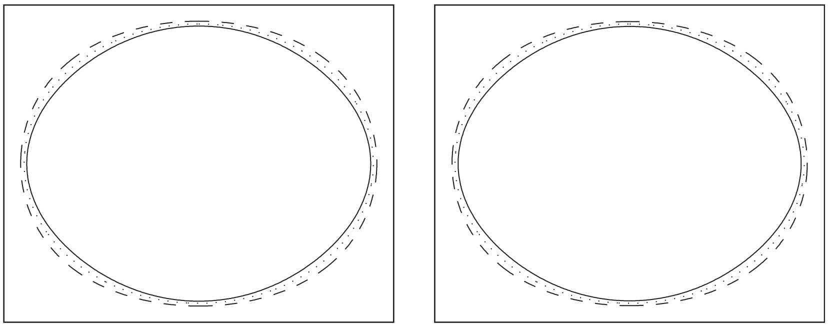

Fig. 4. The approximation curves of an ellipse.

(c) When $s_0 = 2/9$, we have $\lambda = \frac{1}{29}(5\sqrt{893} - 161) \approx -0.3995$,

$$\frac{x^2}{\alpha^2} + \frac{y^2}{\beta^2} - 1 = f(s)(1-t)^2 t^2 \left(t - \frac{1}{3}\right)\left(t - \frac{2}{3}\right),$$
$$-0.1666 < f(s) < -0.1540.$$

For case (a), the approximation curves lie outside the ellipse. For case (b), the approximation curves lie inside the ellipse. The curves in the cases (a) and (b) yield a tight envelope for ellipse (19).

In Fig. 4, ellipses (dotted lines) are shown. In the left figure, for case (a), when $\lambda = -0.42$, the approximation curves $C(t)$ merge with the ellipse. In order to show the difference of the curves, we give curves $C(t)$ with $\lambda = -0.7$ (solid lines) and $\lambda = -0.1$ (dashed lines) in the figure. In the right figure, for case (b), when $\lambda = -0.40$, the approximation curves $C(t)$ merge with the ellipse. In order to show the difference of the curves, we give curves $C(t)$ with $\lambda = -0.7$ (solid lines) and $\lambda = -0.1$ (dashed lines) in Fig. 4.

Also, since $f(s)$ obtains its extremum at $t = 1/2$ and we do not use the exact values of (22) in Fig. 4, in the figure the points of minimum error lie where the ellipse crosses the axes.

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